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Some Fixed Point Theorems By Using Altering Distance Functions

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Abstract: - *In this article, we prove some fixed point theorems in metric space by using altering distance function. Our result are generalization of many previously known results.*

Key words**: -** Metric space, fixed point, Common fixed point, Altering Distance function.
AMS Mathematical Classification:- 47H10, 54H25

1. Introduction and Preliminary

In 1984 , M.S. Khan , M. Swalech and S.Sessa [10] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

Definition 1.1

([10]) . A function $\psi : \Re_+ \to \Re_+$ is called an altering distance function if the following properties are satisfied:

(ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$

(ψ_2) ψ is monotonically non-decreasing.

(ψ_3) ψ is continuous.

By ψ we denote the set of the all altering distance functions.

Theorem1.2

([10]). Let (M, d) be a complete metric space, let $\psi \in \Psi$ and let $S : M \to M$ be a mapping which satisfies the following inequality

$$\psi[d(Sx, Sy)] \le a\psi[d(x, y)]$$

For all $x, y \in M$ and for some $0 < a < 1$. Then S has a unique fixed point $z_0 \in M$ and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Lemma 1.3.

Let (M, d) be a metric space. Let $\{x_n\}$ be a sequence in M such that

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$$

If $\{x_n\}$ is not a Cauchy sequence in M , then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $\{m(k)\}$ and $\{n(k)\}$ with

$$m(k) > n(k) > k$$

Such that

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon_0 \ , \ d(x_{m(k)-1} x_{n(k)}) < \varepsilon_0$$

And

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon_0$

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0$

(iii) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon_0$

Remark 1.4.

Form Lemma 1.3 is easy to get

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon_0$$

Since Banach's fixed point theorem in 1922, because of its simplicity and useful ness, it has become a very popular tool in solving the existence problems in many branches of nonlinear analysis. For some more results of the generalization of this principle.

Beside this, in 1977, Jaggi [7] introduced a new contraction mapping and find a fixed point through rational expression for self mapping, which are following

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx),d(y,Ty)}{d(x,y)}$$

For all $x, y \in X, x \neq y$ *and* $\alpha \in [0,1)$. Then T has a fixed point in X.
The above expression is not valid if $x = y$. This condition is removed by Das and Gupta [5] and proved a fixed point theorem for self mapping on taking following expression,

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx),[1+d(y,Ty)]}{d(x,y)} + \beta d(x,y)$$

For all $x, y \in X, \alpha, \beta \in [0,1),\ 0 < \alpha + \beta < 1$. Then T has a fixed point in X.

In this paper we prove some fixed point and common fixed point theorems for rational expression. Our results is generalization of various known results.

2 Fixed point theorems

Theorem 2.1.

Let (M,d) be a complete metric space, let $\psi \in \Psi$ and let $S: M \to M$ be a mapping which satisfies the following condition:

$$\psi[d(Sx,Sy)] \le a\psi[d(x,y)] + b\psi\left[\frac{d(y,Sy)\ \{1+d(x,Sx)\}}{1+d(x,y)}\right] + c\psi\left[\frac{d(x,Sx).d(y,Sy)}{d(x,y)}\right] \qquad (2.1)$$

For all $x, y \in M$, $x \neq y, a > 0, b > 0, c > 0, a+b+c < 1$

Then S has a unique fixed point $z_0 \in M$, and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof:

Let $x \in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defind as follows:

$$x_{n+1} = Sx_n = S^{n+1}x \text{ for each } n \ge 0.$$

Now

$$\psi[d(x_n, x_{n+1})] = \psi[d(Sx_{n-1}, Sx_n)]$$

$$\le a\psi[d(x_{n-1}, x_n)] + b\psi\left[\frac{d(x_n, Sx_n)\{1+d(x_{n-1}, Sx_{n-1})\}}{1+d(x_{n-1}, x_n)}\right] + c\psi\left[\frac{d(x_{n-1}, Sx_{n-1}).d(x_n, Sx_n)}{d(x_{n-1}, x_n)}\right]$$

$$\le a\psi[d(x_{n-1}, x_n)] + b\psi\left[\frac{d(x_n, x_{n+1})\{1+d(x_{n-1}, x_n)\}}{1+d(x_{n-1}, x_n)}\right] + c\psi\left[\frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}\right]$$

$$\le a\psi[d(x_{n-1}, x_n)] + b\psi\left[d(x_n, x_{n+1})\right] + c\psi[d(x_n, x_{n+1})]$$

$$(1-b-c)\psi[d(x_n, x_{n+1})] \le a\psi[d(x_{n-1}, x_n)]$$

Therefore,

$$\psi[d(x_n, x_{n+1})] \le \frac{a}{1-b-c}\psi[d(x_{n-1}, x_n)]$$

$$\psi[d(x_n, x_{n+1})] \le \left(\frac{a}{1-b-c}\right)^2 \psi[d(x_{n-2}, x_{n-1})]$$

$$\psi[d(x_n,x_{n+1})] \le \left(\frac{a}{1-b-c}\right)^n \psi[d(x_0,x_1)] \text{ ....(2.2)}$$

Since $\frac{a}{1-b-c} \in (0,1)$, form (2.2) we obtain

$$\lim_{n\to\infty} \psi[d(x_n,x_{n+1})] = 0$$

From the fact that $\psi \in \Psi$, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \qquad (2.3)$$

Now , we will show that $\{x_n\}$ is a Cauchy sequence in M . Suppose that $\{x_n\}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon_0 > 0$ such that for each positive integer k , there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) \ge \varepsilon_0 \text{ , } d(x_{m(k)-1}x_{n(k)}) < \varepsilon_0$$

Form Lemma 1.3 and Remark 1.4 we obtain

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \varepsilon_0 \qquad (2.4)$$

$$\lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \varepsilon_0 \qquad (2.5)$$

For $x = x_{m(k)}$ and $y = x_{n(k)}$ from (2.1) we have,

$$\psi[d(x_{m(k)+1},x_{n(k)+1})] = \psi[d(Sx_{m(k)},Sx_{n(k)})]$$

$$\le a\psi[d(x_{m(k)},x_{n(k)})] + b\psi\left[\frac{d(x_{n(k)},Sx_{n(k)})\{1+d(x_{m(k)},Sx_{m(k)})\}}{1+d(x_{m(k)},x_{n(k)})}\right] + c\psi\left[\frac{d(x_{m(k)},Sx_{m(k)}).d(x_{n(k)},Sx_{n(k)})}{d(x_{m(k)},x_{n(k)})}\right]$$

$$\le a\psi[d(x_{m(k)},x_{n(k)})] + b\psi\left[\frac{d(x_{n(k)},x_{n(k)+1})\{1+d(x_{m(k)},x_{m(k)+1})\}}{1+d(x_{m(k)},x_{n(k)})}\right] + c\psi\left[\frac{d(x_{m(k)},x_{m(k)+1}).d(x_{n(k)},x_{n(k)+1})}{d(x_{m(k)},x_{n(k)})}\right]$$

using (2.3), (2.4) and (2.5) we obtain

$$\psi(\varepsilon) = \lim_{k\to\infty} \psi[d(x_{m(k)+1},x_{n(k)+1})]$$

$$\le a \lim_{k\to\infty} \psi[d(x_{m(k)},x_{n(k)})]$$

$$\le a\psi(\varepsilon)$$

Since $a \in (0,1)$, we get a contraction . Then $\{x_n\}$ is a Cauchy sequence in the complete metric space M , thus there exists $z_0 \in M$ such that

$$\lim_{n\to\infty} x_n = z_0$$

Setting $x = x_n$ and $y = z_0$ in (2.1) we have

$$\psi[d(x_{n+1},Sz_0)] = \psi[d(Sx_n,Sz_0)]$$

$$\le a\psi[d(x_n,z_0)] + b\psi\left[\frac{d(z_0,Sz_0)\{1+d(x_n,Sx_n)\}}{1+d(x_n,z_0)}\right] + c\psi\left[\frac{d(x_n,Sx_n).d(z_0,Sz_0)}{d(x_n,z_0)}\right]$$

Therefore,

$$\lim_{n\to\infty} \psi[d(x_{n+1},Sz_0)] \le b\psi[d(z_0,Sz_0)]$$

i.e.,

$$\psi[d(z_0,Sz_0)] \le b\psi[d(z_0,Sz_0)]$$

Since $b \in (0,1)$ then $\psi[d(z_0 , Sz_0)] = 0$ which implies $d(z_0 , Sz_0) = 0$. Thus $z_0 = S z_0$

Now we are going to establish the uniqueness of the fixed point . Let y_0 , z_0 be two fixed points of S such that $y_0 \neq z_0$. Putting $x = y_0$ and $y = z_0$ in (2.1) we get

$$\psi[d(y_0, z_0)] = \psi[d(Sy_0, Sz_0)]$$

$$\leq a\psi[d(y_0, z_0)] + b\psi\left[\frac{d(z_0, Sz_0)\{1+d(y_0, Sy_0)\}}{1+d(y_0, z_0)}\right] + c\psi\left[\frac{d(y_0, Sy_0).d(z_0, Sz_0)}{d(y_0, z_0)}\right]$$

$$\psi[d(y_0, z_0)] \leq a\psi[d(y_0, z_0)]$$

Which implies that $\psi[d(y_0, z_0)] = 0$, so $d(y_0, z_0) = 0$. Thus $y_0 = z_0$

Remarks:

1. In Theorem 2.1, if $b = c = 0$ and $\psi(t) = t$ then we get the result of Banach [1]
2. In Theorem 2.1, if $c = 0$ and $\psi(t) = t$ then we get the result of Das and Gupta [5]
3. In Theorem 2.1, if $a = b = 0$ and $\psi(t) = t$ then we get the result of Jaggi [7]

Corollary 1 .

Let (M, d) be a complete metric space, let $S : M \to M$ be a mapping which satisfies the following condition:

$$d(Sx, Sy) \leq a\, d(x, y) + b\,\frac{d(y, Sy)[1+d(x, Sx)]}{1+d(x, y)} + c\,\frac{d(x, Sx).d(y, Sy)}{d(x, y)}$$

For all $x, y \in M, x \neq y, a > 0, b > 0, c > 0, a+b+c < 1$

Then S has a unique fixed point $z_0 \in M$, and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof :

It is enough if we take $\psi(t) = t$ in theorem 2.1

Corollary 2.

Let (M, d) be a complete metric space, let $S : M \to M$ be a mapping which satisfies the following condition:

$$\int_0^{d(Sx,Sy)} \xi(t)dt \leq a\int_0^{d(x,y)} \xi(t)dt + b\int_0^{\frac{d(y,Sy)[1+d(x,Sx)]}{1+d(x,y)}} \xi(t)dt + c\int_0^{\frac{d(x,Sx).d(y,Sy)}{d(x,y)}} \xi(t)dt$$

For each $x, y \in M, x \neq y, a > 0, b > 0, c > 0, a+b+c < 1$

Where $\xi : R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of R^+ , non-negative and such that for each $\in > 0$, $\int_0^{\in} \xi(t)\,dt > 0$

Then S has a unique fixed point $z_0 \in M$, and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof :

If we take $\psi(t) = \int_0^t \xi(s)ds$ in theorem 2.1 then we get our result

Theorem 2.2.

Let (M, d) be a complete metric space, let $\psi \in \Psi$ and let $S : M \to M$ be a mapping which satisfies the following condition:

$$\psi[d(Sx, Sy)] \leq \alpha\psi[d(x, y)] + \beta\psi\left[\frac{d(x, Sx).d(y, Sy)}{1+d(x, y)}\right] + \gamma\psi\left[\frac{d(x, Sy).d(y, Sx)}{1+d(x, y)}\right]$$

$$+\delta\psi\left[\frac{d(x,Sx).d(x,Sy)}{1+d(x,y)}\right]+\eta\psi\left[\frac{d(y,Sx).d(y,Sy)}{1+d(x,y)}\right] \qquad (2.6)$$

For all $x,y\in M, a>0,\beta>0,\gamma>0,\delta>0,\eta>0,\alpha+\beta+2\delta<1$

Then S has a unique fixed point $z_0\in M$, and moreover for each $x\in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof :

Let $x\in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defind as follows:

$x_{n+1}=Sx_n=S^{n+1}x$ for each $n\geq 0$.

Now

$$\psi[d(x_n,x_{n+1})]=\psi[d(Sx_{n-1},Sx_n)]$$

$$\leq \alpha\psi[d(x_{n-1},x_n)]+\beta\psi\left[\frac{d(x_{n-1},Sx_{n-1}).d(x_n,Sx_n)}{1+d(x_{n-1},x_n)}\right]+\gamma\psi\left[\frac{d(x_{n-1},Sx_n).d(x_n,Sx_{n-1})}{1+d(x_{n-1},x_n)}\right]$$
$$+\delta\psi\left[\frac{d(x_{n-1},Sx_{n-1}).d(x_{n-1},Sx_n)}{1+d(x_{n-1},x_n)}\right]+\eta\psi\left[\frac{d(x_n,Sx_{n-1}).d(x_n,Sx_n)}{1+d(x_{n-1},x_n)}\right]$$
$$\leq \alpha\psi[d(x_{n-1},x_n)]+\beta\psi\left[\frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right]+\gamma\psi\left[\frac{d(x_{n-1},x_{n+1}).d(x_n,x_n)}{1+d(x_{n-1},x_n)}\right]$$
$$+\delta\psi\left[\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_n)}\right]+\eta\psi\left[\frac{d(x_n,x_n).d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right]$$
$$\leq \alpha\psi[d(x_{n-1},x_n)]+\beta\psi\left[\frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right]+\delta\psi\left[\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_n)}\right]$$
$$< \alpha\psi[d(x_{n-1},x_n)]+\beta\psi\left[\frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_{n-1},x_n)}\right]+\delta\psi\left[\frac{d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)}\right]$$
$$< \alpha\psi[d(x_{n-1},x_n)]+\beta\psi[d(x_n,x_{n+1})]+\delta\psi[d(x_{n-1},x_{n+1})]$$
$$\leq \alpha\psi[d(x_{n-1},x_n)]+\beta\psi[d(x_n,x_{n+1})]+\delta\psi[d(x_{n-1},x_n)+d(x_n,x_{n+1})]$$

$$\leq \alpha\psi[d(x_{n-1},x_n)]+\beta\psi[d(x_n,x_{n+1})]+\delta\psi[d(x_{n-1},x_n)]+\delta\psi[d(x_n,x_{n+1})]$$
$$(1-\beta-\delta)\psi[d(x_n,x_{n+1})]\leq(\alpha+\delta)\psi[d(x_{n-1},x_n)]$$

Therefore,

$$\psi[d(x_n,x_{n+1})]\leq\frac{\alpha+\delta}{1-\beta-\delta}\psi[d(x_{n-1},x_n)]$$

Similarly,

$$\psi[d(x_{n-1},x_n)]\leq\frac{\alpha+\delta}{1-\beta-\delta}\psi[d(x_{n-2},x_{n-1})]$$

And

$$\psi[d(x_n,x_{n+1})]\leq\left(\frac{\alpha+\delta}{1-\beta-\delta}\right)^2\psi[d(x_{n-2},x_{n-1})]$$

Continuing this process, we get in general

$$\psi[d(x_n,x_{n+1})]\le\left(\frac{\alpha+\delta}{1-\beta-\delta}\right)^n\psi[d(x_0,x_1)] \qquad (2.7)$$

Since $\left(\frac{\alpha+\delta}{1-\beta-\delta}\right)\in(0,1)$, from (2.7) we obtain

$$\lim_{n\to\infty}\psi[d(x_n,x_{n+1})]=0$$

From the fact that $\psi\in\Psi$, we have

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0 \qquad (2.8)$$

Now , we will show that $\{x_n\}$ is a Cauchy sequence in M . Suppose that $\{x_n\}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon_0>0$ such that for each positive integer k , there are positive integers $m(k)$ and $n(k)$ with $m(k)>n(k)>k$ such that

$$d(x_{m(k)},x_{n(k)})\ge\varepsilon_0\ ,\ d(x_{m(k)-1},x_{n(k)})<\varepsilon_0$$

Form Lemma 1.3 and Remark 1.4 we obtain

$$\lim_{n\to\infty}d(x_{m(k)},x_{n(k)})=\varepsilon_0 \qquad (2.9)$$

$$\lim_{n\to\infty}d(x_{m(k)+1},x_{n(k)+1})=\varepsilon_0 \qquad (2.10)$$

For $x=x_{m(k)}$ and $y=x_{n(k)}$ from (2.6) we have,

$$\psi[d(x_{m(k)+1},x_{n(k)+1})]=\psi[d(Sx_{m(k)},Sx_{n(k)})]$$

$$\le\alpha\psi[d(x_{m(k)},x_{n(k)})]+\beta\psi\left[\frac{d(x_{m(k)},Sx_{m(k)}).d(x_{n(k)},Sx_{n(k)})}{1+d(x_{m(k)},x_{n(k)})}\right]+\gamma\psi\left[\frac{d(x_{m(k)},Sx_{n(k)}).d(x_{n(k)},Sx_{m(k)})}{1+d(x_{m(k)},x_{n(k)})}\right]$$

$$+\delta\psi\left[\frac{d(x_{m(k)},Sx_{m(k)}).d(x_{m(k)},Sx_{n(k)})}{1+d(x_{m(k)},x_{n(k)})}\right]+\eta\psi\left[\frac{d(x_{n(k)},Sx_{m(k)}).d(x_{n(k)},Sx_{n(k)})}{1+d(x_{m(k)},x_{n(k)})}\right]$$

$$\le\alpha\psi[d(x_{m(k)},x_{n(k)})]+\beta\psi\left[\frac{d(x_{m(k)},x_{m(k)+1}).d(x_{n(k)},x_{n(k)+1})}{1+d(x_{m(k)},x_{n(k)})}\right]+\gamma\psi\left[\frac{d(x_{m(k)},x_{n(k)+1}).d(x_{n(k)},x_{m(k)+1})}{1+d(x_{m(k)},x_{n(k)})}\right]$$

$$+\delta\psi\left[\frac{d(x_{m(k)},x_{m(k)+1}).d(x_{m(k)},x_{n(k)+1})}{1+d(x_{m(k)},x_{n(k)})}\right]+\eta\psi\left[\frac{d(x_{n(k)},x_{m(k)+1}).d(x_{n(k)},x_{n(k)+1})}{1+d(x_{m(k)},x_{n(k)})}\right]$$

Using (2.8), (2.9) and (2.10) we obtain,

$$\psi(\varepsilon)=\lim_{k\to\infty}\psi[d(x_{m(k)+1},x_{n(k)+1})]$$

$$\le\alpha\lim_{n\to\infty}\psi[d(x_{m(k)},x_{n(k)})]+\gamma\lim_{n\to\infty}\psi\left[\frac{d(x_{m(k)},x_{n(k)+1}).d(x_{n(k)},x_{m(k)+1})}{1+d(x_{m(k)},x_{n(k)})}\right]$$

$$\le \alpha \lim_{n\to\infty} \psi[d(x_{m(k)}, x_{n(k)})] + \gamma \lim_{n\to\infty} \psi\left[\frac{\{d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})\}.\{d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1})\}}{1+d(x_{m(k)}, x_{n(k)})}\right]$$

$$\le \alpha\psi(\varepsilon) + \gamma\psi\left(\frac{\varepsilon.\varepsilon}{1+\varepsilon}\right)$$

Since ε be an arbitrary

$$\le \alpha\psi(\varepsilon) + \gamma\psi(\varepsilon)$$

$$\psi(\varepsilon) \le (\alpha+\gamma)\psi(\varepsilon)$$

Since $\alpha, \gamma \in (0,1)$ and $\alpha+\gamma<1$ we get a contraction. Then $\{x_n\}$ is a Cauchy sequence in the complete metric space M , thus there exists $z_0 \in M$ such that

$$\lim_{n\to\infty} x_n = z_0$$

Setting $x = x_n$ and $y = z_0$ in (2.6) we have

$$\psi[d(x_{n+1}, Sz_0)] = \psi[d(Sx_n, Sz_0)]$$

$$\le \alpha\psi[d(x_n, z_0)] + \beta\psi\left[\frac{d(x_n, Sx_n).d(z_0, Sz_0)}{1+d(x_n, z_0)}\right] + \gamma\psi\left[\frac{d(x_n, Sz_0).d(z_0, Sx_n)}{1+d(x_n, z_0)}\right]$$

$$+\delta\psi\left[\frac{d(x_n, Sx_n).d(x_n, Sz_0)}{1+d(x_n, z_0)}\right] + \eta\psi\left[\frac{d(z_0, Sx_n).d(z_0, Sz_0)}{1+d(x_n, z_0)}\right]$$

Therefore,

$$\lim_{n\to\infty} \psi[d(x_{n+1}, Sz_0)] \le 0$$

$$\psi[d(z_0, Sz_0)] \le 0$$

Since ψ is continuous non- decreasing which implies $d(z_0, Sz_0) \le 0$ Which is contraction.

$i.e.\ z_0 = Sz_0$

Thus z_0 is fixed point of S .

Now we are going to establish the uniqueness of the fixed point . Let y_0 , z_0 be two fixed points of S such that $y_0 \ne z_0$. Putting $x = y_0$ and $y = z_0$ in (2.6) we get

$$\psi[d(y_0, z_0)] = \psi[d(Sy_0, Sz_0)]$$

$$\le \alpha\psi[d(y_0, z_0)] + \beta\psi\left[\frac{d(y_0, Sy_0).d(z_0, Sz_0)}{1+d(y_0, z_0)}\right] + \gamma\psi\left[\frac{d(y_0, Sz_0).d(z_0, Sy_0)}{1+d(y_0, z_0)}\right]$$

$$+\delta\psi\left[\frac{d(y_0, Sy_0).d(y_0, Sz_0)}{1+d(y_0, z_0)}\right] + \eta\psi\left[\frac{d(z_o, Sy_0).d(z_0, Sz_0)}{1+d(y_0, z_0)}\right]$$

$$\le \alpha\psi[d(y_0, z_0)] + \gamma\psi\left[\frac{d(y_0, z_0).d(z_0, y_0)}{d(y_0, z_0)}\right]$$

$$\le \alpha\psi[d(y_0, z_0)] + \gamma\psi[d(y_0, z_0)]$$

$$\psi[d(y_0, z_0)] \le (\alpha+\gamma)\psi[d(y_0, z_0)]$$

Which implies that $\psi[d(y_0, z_0)] = 0$,so $d(y_0, z_0) = 0$. Thus $y_0 = z_0$

Corollary 3:

Let (M,d) be a complete metric space, let $S:M\to M$ be a mapping which satisfies the following condition:

$$d(Sx,Sy)\le \alpha d(x,y)+\beta\frac{d(x,Sx).d(y,Sy)}{1+d(x,y)}+\gamma\frac{d(x,Sy).d(y,Sx)}{1+d(x,y)}$$
$$+\delta\frac{d(x,Sx).d(x,Sy)}{1+d(x,y)}+\eta\frac{d(y,Sx).d(y,Sy)}{1+d(x,y)}$$

For all $x,y\in M, a>0,\beta>0,\gamma>0,\delta>0,\eta>0,\alpha+\beta+2\delta<1$

Then S has a unique fixed point $z_0\in M$, and moreover for each $x\in M$ $\lim_{n\to\infty}S^n x=z_0$

Proof :

It is enough if we take $\psi(t)=t$ in theorem 2.6

Corollary 4:

Let (M,d) be a complete metric space, and let $S:M\to M$ be a mapping which satisfies the following condition:

$$\int_0^{d(Sx,Sy)}\xi(t)dt\le\alpha\int_0^{d(x,y)}\xi(t)dt+\beta\int_0^{\frac{d(x,Sx).d(y,Sy)}{1+d(x,y)}}\xi(t)dt+\gamma\int_0^{\frac{d(x,Sy).d(y,Sx)}{1+d(x,y)}}\xi(t)dt$$
$$+\delta\int_o^{\frac{d(x,Sx).d(x,Sy)}{1+d(x,y)}}\xi(t)dt+\eta\int_o^{\frac{d(y,Sx).d(y,Sy)}{1+d(x,y)}}\xi(t)dt$$

For all $x,y\in M, a>0,\beta>0,\gamma>0,\delta>0,\eta>0,\alpha+\beta+2\delta<1$

Where $\xi:R^+\to R^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of R^+ ,non-negative and such that for each $\in>0$, $\int_0^{\in}\xi(t)\,dt>0$

Then S has a unique fixed point $z_0\in M$, and moreover for each $x\in M$ $\lim_{n\to\infty}S^n x=z_0$

Proof :

If we take $\psi(t)=\int_0^t\xi(s)ds$ an in Theorem 2.6 then we get our result

Theorem 2.3.

Let (M,d) be a complete metric space, let $\psi\in\Psi$ and let $S:M\to M$ be a mapping which satisfies the following condition:

$$\psi[d(Sx,Sy)]\le\alpha\psi\left[\frac{d^2(x,Sx)+d^2(y,Sy)}{d(x,Sx)+d(y,Sy)}\right]+\beta\psi\left[\frac{d^2(x,Sy)+d^2(y,Sx)}{d(x,Sy)+d(y,Sx)}\right] \qquad (2.11)$$

For all $x,y\in M,\alpha>0,\beta>0,\alpha+\beta<\frac{1}{2}$

Then S has a unique fixed point $z_0\in M$, and moreover for each $x\in M$ $\lim_{n\to\infty}S^n x=z_0$

Proof :

Let $x\in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defind as follows:

$x_{n+1}=Sx_n=S^{n+1}x$ for each $n\ge0$.

Now

$$\psi[d(x_n,x_{n+1})]=\psi[d(Sx_{n-1},Sx_n)]$$
$$\le\alpha\psi\left[\frac{d^2(x_{n-1},Sx_{n-1})+d^2(x_n,Sx_n)}{d(x_{n-1},Sx_{n-1})+d(x_n,Sx_n)}\right]+\beta\psi\left[\frac{d^2(x_{n-1},Sx_n)+d^2(x_n,Sx_{n-1})}{d(x_{n-1},Sx_n)+d(x_n,Sx_{n-1})}\right]$$

$$\leq \alpha\psi\left[\frac{d^2(x_{n-1},x_n)+d^2(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})}\right]+\beta\psi\left[\frac{d^2(x_{n-1},x_{n+1})+d^2(x_n,x_n)}{d(x_{n-1},x_{n+1})+d(x_n,x_n)}\right]$$

$$\leq \alpha\psi\left[\frac{\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}^2-2d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})}\right]+\beta\psi\left[\frac{d^2(x_{n-1},x_{n+1})}{d(x_{n-1},x_{n+1})}\right]$$

$$\leq \alpha\psi\left[\frac{\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}^2}{d(x_{n-1},x_n)+d(x_n,x_{n+1})}\right]+\beta\psi\left[\frac{d^2(x_{n-1},x_{n+1})}{d(x_{n-1},x_{n+1})}\right]$$

$$\leq \alpha\psi[d(x_{n-1},x_n)+d(x_n,x_{n+1})]+\beta\psi[d(x_{n-1},x_{n+1})]$$

$$(1-\alpha-\beta)\psi[d(x_n,x_{n+1})]\leq(\alpha+\beta)\psi[d(x_{n-1},x_n)]$$

Therefore,

$$\psi[d(x_n,x_{n+1})]\leq\frac{\alpha+\beta}{1-\alpha-\beta}\psi[d(x_{n-1},x_n)]$$

Similarly,

$$\psi[d(x_{n-1},x_n)]\leq\frac{\alpha+\beta}{1-\alpha-\beta}\psi[d(x_{n-2},x_{n-1})]$$

And

$$\psi[d(x_n,x_{n+1})]\leq\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)^2\psi[d(x_{n-2},x_{n-1})]$$

Continuing this process, we get in general

$$\psi[d(x_n,x_{n+1})]\leq\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)^n\psi[d(x_0,x_1)] \qquad (2.12)$$

Since $\left(\frac{\alpha+\beta}{1-\alpha-\beta}\right)\in(0,1)$, from (2.12) we have

$$\lim_{n\to\infty}\psi[d(x_n,x_{n+1})]=0$$

From the fact that $\psi\in\Psi$, we have

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0 \qquad (2.13)$$

Now we will show that $\{x_n\}$ is a Cauchy sequence in M . Suppose that $\{x_n\}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon_0>0$ such that for each positive integer k , there are positive integers $m(k)$ and $n(k)$ with $m(k)>n(k)>k$ such that

$$d(x_{m(k)},x_{n(k)})\geq\varepsilon_0\ ,\ d(x_{m(k)-1}x_{n(k)})<\varepsilon_0$$

Form Lemma 1.3 and Remark 1.4 we obtain

$$\lim_{k\to\infty}d(x_{m(k)},x_{n(k)})=\varepsilon_0 \qquad (2.14)$$

$$\lim_{k\to\infty}d(x_{m(k)+1},x_{n(k)+1})=\varepsilon_0 \qquad (2.15)$$

For $x=x_{m(k)}$ and $x=x_{n(k)}$ from (2.11) we have,

$$\psi[d(x_{m(k)+1},x_{n(k)+1})]=\psi[d(Sx_{m(k)},Sx_{n(k)})]$$

$$\leq \alpha\psi\left[\frac{d^2(x_{m(k)},Sx_{m(k)})+d^2(x_{n(k)},Sx_{n(k)})}{d(x_{m(k)},Sx_{m(k)})+d(x_{n(k)},Sx_{n(k)})}\right]+\beta\psi\left[\frac{d^2(x_{m(k)},Sx_{n(k)})+d^2(x_{n(k)},Sx_{m(k)})}{d(x_{m(k)},Sx_{n(k)})+d(x_{n(k)},Sx_{m(k)})}\right]$$

$$\leq \alpha\psi\left[\frac{d^2(x_{m(k)},x_{m(k)+1})+d^2(x_{n(k)},x_{n(k)+1})}{d(x_{m(k)},x_{m(k)+1})+d(x_{n(k)},x_{n(k)+1})}\right]+\beta\psi\left[\frac{d^2(x_{m(k)},x_{n(k)+1})+d^2(x_{n(k)},x_{m(k)+1})}{d(x_{m(k)},x_{n(k)+1})+d(x_{n(k)},x_{m(k)+1})}\right]$$

Using (2.13) , (2.14) and (2.15) we obtain

$$\psi(\varepsilon)=\lim_{k\to\infty}\psi[d(x_{m(k)+1},x_{n(k)+1})]$$
$$\leq \beta\lim_{k\to\infty}\psi[d(x_{m(k)},x_{n(k)+1})+d(x_{n(k)},x_{m(k)+1})]$$

From lemma 1.3 we have

$$\psi(\varepsilon)\leq\beta\psi(\varepsilon)$$

Since $\beta\in(0,1)$, we get a contraction . Then $\{x_n\}$ is a Cauchy sequence in the complete metric space M , thus there exists $z_0\in M$ such that $\lim_{n\to\infty}x_n=z_0$

Setting $x=x_n$ and $y=z_0$ in (2.11) we have

$$\psi[d(x_{n+1},Sz_0)]=\psi[d(Sx_n,Sz_0)]$$
$$\leq\alpha\psi\left[\frac{d^2(x_n,Sx_n)+d^2(z_o,Sz_0)}{d(x_n,Sx_n)+d(z_0,Sz_0)}\right]+\beta\psi\left[\frac{d^2(x_n,Sz_0)+d^2(z_0,Sx_n)}{d(x_n,Sz_0)+d(z_0,Sx_n)}\right]$$

Therefore,

$$\lim_{n\to\infty}\psi[d(x_{n+1},Sz_0)]\leq(\alpha+\beta)\psi[d(z_0,Sz_0)]$$
$$\lim_{n\to\infty}\psi[d(z_o,Sz_0)]\leq(\alpha+\beta)\psi[d(z_0,Sz_0)]$$

Since $\alpha,\beta\in(0,1)$ and $(\alpha+\beta)<\frac{1}{2}$ then $\psi[d(z_0\,,Sz_0)]=0$ which implies that $d(z_0\,,Sz_0)=0$.

Thus $z_0=Sz_0$

Now we are going to establish the uniqueness of the fixed point. Let y_0 , z_0 be two fixed points of S such that $y_0\neq z_0$. Putting $x=y_0$ and $y=z_0$ in (2.11) we get

$$\psi[d(y_0,z_0)]\leq\psi[d(Sy_0,Sz_0)]$$
$$\leq\alpha\psi\left[\frac{d^2(y_0,Sy_0)+d^2(z_0,Sz_0)}{d(y_0,Sy_0)+d(z_0,Sz_0)}\right]+\beta\psi\left[\frac{d^2(y_0,Sz_0)+d^2(z_0,Sy_0)}{d(y_0,Sz_0)+d(z_0,Sy_0)}\right]$$
$$\leq\beta\psi\left[\frac{d^2(y_0,z_0)+d^2(z_0,y_0)}{d(y_0,z_0)+d(z_0,y_0)}\right]$$
$$\leq\beta\psi[d(y_0,z_0)+d(z_0,y_0)]$$
$$\psi[d(y_0,z_0)]\leq2\beta\psi[d(y_0,z_0)]$$

Which implies that $\psi[d(y_0,z_0)]=0$,so $d(y_0,z_0)=0$. Thus $y_0=z_0$

Corollary 5: Let (M,d) be a complete metric space, let $S:M\to M$ be a mapping which satisfies the following condition:

$$d(Sx,Sy)\leq\alpha\frac{d^2(x,Sx)+d^2(y,Sy)}{d(x,Sx)+d(y,Sy)}+\beta\frac{d^2(x,Sy)+d^2(y,Sx)}{d(x,Sy)+d(y,Sx)}$$

For all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < \frac{1}{2}$

Then S has a unique fixed point $z_0 \in M$, and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof :

It is enough if we take $\psi(t) = t$ in theorem 2.11

Corollary 6 .

Let (M, d) be a complete metric space, let $S : M \to M$ be a mapping which satisfies the following condition:

$$\int_0^{d(Sx,Sy)} \xi(t)dt \le \alpha \int_0^{\frac{d^2(x,Sx)+d^2(y,Sy)}{d(x,Sx)+d(y,Sy)}} \xi(t)dt + \beta \int_0^{\frac{d^2(x,Sy)+d^2(y,Sx)}{d(x,Sy)+d(y,Sx)}} \xi(t)dt$$

For all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < \frac{1}{2}$

Where $\xi : R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of R^+ , non-negative and such that for each $\in > 0$, $\int_0^{\in} \xi(t)\, dt > 0$

Then S has a unique fixed point $z_0 \in M$, and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Proof :

If we take $\psi(t) = \int_0^t \xi(s)ds$ an in theorem 2.11 then we get our result

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